ON A DIFFERENTIAL INEQUALITY OF CESARI AND TURNER

Richard Bellman

P-1233

December 16, 1957

The RAND Corporation
1700 MAIN ST. · SANTA MONICA · CALIFORNIA

SUMMARY

This paper contains a simplified proof, and indicated generalization, of a result of Cesari and Turner, "On a Lemma in the Direct Method of the Calculus of Variations."

SUMMARY

This paper contains a simplified proof, and indicated generalization, of a result of Cesari and Turner, "On a Lemma in the Direct Method of the Calculus of Variations."

ON A DIFFERENTIAL INEQUALITY OF CESARI AND TURNER

Richard Bellman

1. Introduction

In a recent paper, [1], Cesari and Turner give a proof of the following

Lemma. *If* $f(x)$, $0 \leq x \leq a$, *is any nonnegative* L^n*-integrable function with*

$$(1) \qquad kf^m(x) \geq (a - x)^p \int_x^a (a - u)^q f^n(u)du > 0,$$

for almost all x *in* $[0,a]$, *and, in particular, for* $x = 0$, *where* $k, m, n, p, q > 0$ *are given constants with* $m > n$, *then*

$$(2) \qquad a \leq cf^r(0),$$

where c *and* r *are constants depending only on* $k, m, n, p,$ *and* q.

The authors give a very ingenious proof of this result, which is not quite direct. Here we shall show that some preliminary transformations enable us to present a simple, direct proof, and indicate some generalizations.

2. Proof of Lemma

Let us begin with the transformation

$$(1) \qquad f(x) = (a - x)^{p/m} w(x),$$

which converts (1.1) into an inequality of the form

(2) $$k\, w(x)^m \geq \int_x^a (a-u)w^n(u)du.$$

Now replace $w^n(u)$ by $z(u)$, and then x by $a-x$. If we set $y(x) = z(a-x)$ for $0 \leq x \leq a$, we obtain a new inequality of the form

(3) $$k\, y(x)^c \geq \int_0^x y(u)u^b du,$$

where $c > 1$.

The change of independent variable $x^{b+1} = r$, and the further change of dependent variable $y(x^{1/(b+1)}) = g(x)$, yields the simplified inequality

(4) $$k_1\, g(x)^c \geq \int_0^x g(r)dr,$$

valid for almost all x in $[0,a]$, and valid for $x = a$, with $c > 1$ and $g(x) \geq 0$.

From this we wish to conclude that

(5) $$k_2\, g(a)^{c_1} \geq a,$$

where c_1 depends on c, and k_2 on k_1.

This is easily done. Let $h(x) = \int_0^x g(r)dr$. Then (4) is equivalent to

(6) $$h^{-1/c} \frac{dh}{dx} \geq k_3.$$

Integrating both sides between 0 and x, we obtain

(7) $$h^{(1-1/c)}(x) \geq k_3 x,$$

since $h(0) = 0$. Setting $x = a$, we have

(8) $$h(a) \geq k_4 a^{c/(c-1)}.$$

Combining (4) and (8), we have

(9) $$k_1 \, g(a)^c \geq h(a) \geq k_4 \, a^{c/(c-1)},$$

which yields the desired result.

3. Discussion

The same techniques can be used to obtain a corresponding inequality for the case where we start with a relation

(1) $$F(f(x)) \geq G(a-x) \int_x^a H(a-u)K(f(u))du,$$

under appropriate assumptions concerning the functions F, G, H, and K.

The RAND Corporation
Santa Monica, California

REFERENCES

1. Cesari, L., and L. H. Turner, "On a Lemma in the Direct Method of the Calculus of Variations," Rendiconti del Circolo Matematico di Palermo, Serie II-Tomo VI, 1957, pp. 109-113.